Higher-Degree Equations

Teacher guide and model answers

Chapter 3 | Module 14 | Student Book pages 142-149 | Suggested time: 50 minutes | 20 marks

Learning goals

- Explain and apply degree and number of roots with valid conditions and clear reasoning.
- Explain and apply cubic from roots with valid conditions and clear reasoning.
- Explain and apply repeated root with valid conditions and clear reasoning.

Quick reference

Degree and number of roots	A degree-n polynomial has n complex roots counted with multiplicity.
Cubic from roots	A cubic with roots r,s,t is proportional to (x-r)(x-s)(x-t).
Repeated root	A repeated root is shared by P(x) and its derivative P'(x).

Worked example

To solve x^3-4x^2+x+6=0, verify P(2)=0, then divide by x-2 to get (x-2)(x-3)(x+1)=0; the roots are 2, 3 and -1.

Graduated practice

Answer without viewing the key. Show complete working for short-response questions.

1. How many roots does a degree-4 polynomial have over the complex numbers, counting multiplicity? [1 marks]

A) 2 B) 3

C) 4 D) It cannot be determined

Answer: 4

The Fundamental Theorem of Algebra gives a degree-n polynomial n complex roots when multiplicity is counted.

2. If 2 is a root of $x^3+kx^2-4x-4=0$, find k. [1 marks]

A) -1 B) 0

C) 1 D) 2

Answer: 1

Substitute x=2: 8+4k-8-4=0, so 4k-4=0 and k=1.

3. Which monic cubic equation has roots -1, 2 and 3? [1 marks]

A) $x^3-4x^2+x+6=0$ B) $x^3+4x^2-x-6=0$

C) $x^3-6x^2+x+4=0$ D) $x^3-4x^2-x-6=0$

Answer: $x^3-4x^2+x+6=0$

The equation is (x+1)(x-2)(x-3)=0, which expands to $x^3-4x^2+x+6=0$.

4. If ω is a nonreal cube root of unity, what is ω^{2026}? [1 marks]

A) 1 B) ω

C) ω^2 D) -1

Answer: ω

Powers repeat every 3 because $\omega^3=1$. Since 2026 has remainder 1 on division by 3, $\omega^{2026}=\omega$.

5. In $(x-2)^2(x+1)=0$, which root is repeated? [1 marks]

A) -1 with multiplicity 2 B) 1 with multiplicity 2

C) 2 with multiplicity 2 D) 2 with multiplicity 3

Answer: 2 with multiplicity 2

The factor x-2 occurs to the second power, so root 2 has multiplicity 2.

6. What is the real solution set of $x^4-5x^2+4=0$? [1 marks]

A) {-2, 2}
B) {-1, 1}
C) {-2, -1, 1, 2}
D) {1, 2}

Answer: {-2, -1, 1, 2}

Let $y=x^2$. Then $y^2-5y+4=(y-1)(y-4)$, so $x^2=1$ or $x^2=4$.

7. True or false: every degree-n polynomial equation has n distinct real roots. [1 marks]

□ True □ False

Answer: False

The count n applies over the complex numbers and includes multiplicity; roots may be nonreal or repeated.

8. True or false: if r is a repeated root of P(x), then P(r)=0 and P'(r)=0. [1 marks]

□ True □ False

Answer: True

The factor $(x-r)^2$ divides P(x), so P and its derivative share the root r.

9. Solve $x^3-4x^2+x+6=0$ using a known integer root and degree reduction. [3 marks]

Answer: x=-1, x=2 or x=3

P(2)=0. Dividing by x-2 gives $x^2-2x-3=(x-3)(x+1)$, so the roots are 2, 3 and -1.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

10. Form the monic cubic equation whose roots are 1, -2 and 4. [3 marks]

Answer: $x^3-3x^2-6x+8=0$

Write $(x-1)(x+2)(x-4)=0$ and expand to obtain $x^3-3x^2-6x+8=0$.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

11. If $\omega^3=1$ and $1+\omega+\omega^2=0$, simplify $(1+\omega)^6$. [3 marks]

Answer: 1

Since $1+\omega=-\omega^2$, $(1+\omega)^6=(-\omega^2)^6=\omega^{12}=(\omega^3)^4=1$.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

12. If 1 is a repeated root of $P(x)=x^3+ax^2+bx+2$, find a and b, then factor $P(x)$. [3 marks]

Answer: $a=0$, $b=-3$, and $P(x)=(x-1)^2(x+2)$

$P(1)=0$ gives $a+b=-3$, while $P'(1)=0$ gives $2a+b=-3$. Hence $a=0$ and $b=-3$, and $x^3-3x+2=(x-1)^2(x+2)$.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

Challenge and review

1. Derive Vieta's relations for a monic cubic with roots r, s and t by expanding (x-r)(x-s)(x-t).

Expansion gives $x^3-(r+s+t)x^2+(rs+rt+st)x-rst$. Thus the root sum is the negative x^2 coefficient, the pairwise-product sum is the x coefficient, and the triple product is the negative constant term.

2. Solve $x^6-5x^3+4=0$ over the complex numbers, listing all six roots using ω.

Let $y=x^3$, giving $(y-1)(y-4)=0$. From $x^3=1$ the roots are 1, ω and ω^2. If a=□4, the roots of $x^3=4$ are a, aω and $a\omega^2$. These are all six roots.

3. Error analysis: when r is a root, a student writes the factor x+r. Correct the claim and state when x+r would be the right factor.

If r is a root, then $P(r)=0$ and the factor is $x-r$. The factor $x+r$ is correct when the root is $-r$, since $x-(-r)=x+r$.

Follow-up plan

Below 14/20: revisit root counting with multiplicity, degree reduction and cube roots of unity, then retry Questions 2-6 and 9-12. At 14 or above: continue to the online module challenge.