Polynomial Theorems

Teacher guide and model answers

Chapter 3 | Module 13 | Student Book pages 132-141 | Suggested time: 50 minutes | 20 marks

Learning goals

- Explain and apply remainder theorem with valid conditions and clear reasoning.
- Explain and apply unknown coefficients with valid conditions and clear reasoning.
- Explain and apply factor theorem with valid conditions and clear reasoning.

Quick reference

Remainder theorem	The remainder when P(x) is divided by x-a is P(a).
Unknown coefficients	Remainder conditions become equations in unknown polynomial coefficients.
Factor theorem	x-a is a factor of P(x) exactly when P(a)=0.

Worked example

For the remainder when P(x)=2x^3-3x+4 is divided by x-2, substitute x=2 directly: P(2)=16-6+4=14.

Graduated practice

Answer without viewing the key. Show complete working for short-response questions.

1. Find the remainder when P(x)=2x^3-3x+4 is divided by x-2. [1 marks]

A) 6 B) 10

C) 14 D) 18

Answer: 14

By the Remainder Theorem, substitute x=2: P(2)=2×8-3×2+4=14.

2. Find the remainder when P(x)=4x^2+2x-1 is divided by 2x+3. [1 marks]

A) -5 B) 1

C) 5 D) 11

Answer: 5

The divisor 2x+3 has root -3/2, so P(-3/2)=9-3-1=5.

3. The remainder when P(x)=x^3+kx+2 is divided by x-1 is 7. Find k. [1 marks]

A) 2 B) 3

C) 4 D) 6

Answer: 4

P(1)=1+k+2=7, so k=4.

4. What is the quotient when x^3-4x^2+x+6 is divided by x-2? [1 marks]

A) x^2-2x-3 B) x^2-6x-3

C) x^2+2x-3 D) x^2-2x+3

Answer: x^2-2x-3

Synthetic division by 2 gives quotient coefficients 1, -2, -3 and remainder zero.

5. Which expression is a factor of P(x)=x^3-4x? [1 marks]

A) x-1 B) x-2

C) x-3 D) x-4

Answer: x-2

P(2)=8-8=0, so x-2 is a factor by the Factor Theorem.

6. What is the complete factorisation of x^3-6x^2+11x-6? [1 marks]

A) (x+1)(x+2)(x+3) B) (x-1)(x-2)(x-3)

C) (x-1)(x+2)(x-3) D) (x+1)(x-2)(x+3)

Answer: (x-1)(x-2)(x-3)

The values 1, 2 and 3 are zeros, and the corresponding product expands to the original polynomial.

7. True or false: when using synthetic division on x^3+5 by x-2, the coefficients of x^2 and x must be entered as zeros. [1 marks]

□ True □ False

Answer: True

The coefficient list is 1, 0, 0, 5; omitting missing powers shifts the coefficient positions.

8. True or false: if P(a)=0, then x+a is a factor of P(x). [1 marks]

□ True □ False

Answer: False

The Factor Theorem gives the factor x-a when P(a)=0, not x+a.

9. If x-1 and x+2 are factors of P(x)=x^3+ax^2+bx+6, find a and b. [3 marks]

Answer: a=-2, b=-5

P(1)=0 gives a+b=-7, while P(-2)=0 gives 2a-b=1. Solving gives a=-2 and b=-5.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

10. Divide 2x^3+3x^2-8x-12 by x+2 using synthetic division. State the quotient and remainder. [3 marks]

Answer: Quotient 2x^2-x-6; remainder 0

Use -2 with coefficients 2, 3, -8, -12 to obtain 2, -1, -6, 0.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

11. Solve 2x^3-3x^2-8x+12=0 using the Factor Theorem and degree reduction. [3 marks]

Answer: x=-2, x=3/2 or x=2

P(2)=0. Dividing by x-2 gives 2x^2+x-6=(2x-3)(x+2), so the roots are 2, 3/2 and -2.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

12. When P(x) is divided by (x-1)(x+2), P(1)=3 and P(-2)=-6. Find the remainder. [3 marks]

Answer: 3x

The remainder has degree below 2, so let R(x)=ax+b. From a+b=3 and -2a+b=-6, a=3 and b=0; therefore R(x)=3x.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

Challenge and review

1. Prove that P(x)-P(a) is divisible by x-a for every polynomial P, then deduce the Remainder Theorem.

For positive n, x^n-a^n=(x-a)(x^(n-1)+x^(n-2)a+...+a^(n-1)). Applying this termwise shows P(x)-P(a) is a multiple of x-a, so P(x)=(x-a)Q(x)+P(a) and the remainder is P(a).

2. Factor x^4-5x^2+4 completely using the Factor Theorem and degree reduction.

Let y=x^2. Then y^2-5y+4=(y-1)(y-4), so x^4-5x^2+4=(x^2-1)(x^2-4)=(x-1)(x+1)(x-2)(x+2).

3. Error analysis: to find a remainder on division by 2x-1, a student substitutes x=2. Identify the correct substitution and explain where it comes from.

Solve the divisor equation 2x-1=0 to get x=1/2. The remainder is therefore P(1/2). Substituting 2 would correspond to x-2, not 2x-1.

Follow-up plan

Below 14/20: revisit the divisor root, zero coefficients, and the Remainder and Factor Theorems, then retry Questions 2-6 and 9-12. At 14 or above: continue to the online module challenge.