Quadratic Equations

Teacher guide and model answers

Chapter 3 | Module 12 | Student Book pages 115-131 | Suggested time: 50 minutes | 20 marks

Learning goals

- Explain and apply quadratic formula with valid conditions and clear reasoning.
- Explain and apply roots and coefficients with valid conditions and clear reasoning.
- Explain and apply signs of roots with valid conditions and clear reasoning.

Quick reference

Quadratic formula	For $ax^2+bx+c=0$ with $a\neq0$, $x=(-b\pm\sqrt{(b^2-4ac)})/(2a)$.
Roots and coefficients	For $ax^2+bx+c=0$, $\alpha+\beta=-b/a$ and $\alpha\beta=c/a$.
Signs of roots	The signs of real roots can be inferred from their product, sum, and discriminant.

Worked example

To solve $2x^2+x-3=0$: $D=25$, so the quadratic formula gives $x=(-1\pm5)/4$; hence $x=1$ or $x=-3/2$.

Graduated practice

Answer without viewing the key. Show complete working for short-response questions.

1. What is the solution set of $x^2-5x+6=0$? [1 marks]

A) {2, 3}

B) {-2, -3}

C) {1, 6}

D) {5, 6}

Answer: {2, 3}

Factor: $x^2-5x+6=(x-2)(x-3)$, so $x=2$ or $x=3$.

2. What is the nature of the roots of $2x^2+4x+5=0$ over the real numbers? [1 marks]

A) Two equal real roots

B) Two distinct real roots

C) No real roots

D) One real root equal to -1

Answer: No real roots

The discriminant is $D=4^2-4\times2\times5=-24<0$, so there are no real roots.

3. If α and β are the roots of $3x^2-7x-6=0$, find $\alpha+\beta$ and $\alpha\beta$. [1 marks]

A) $\alpha+\beta=-7/3$, $\alpha\beta=-2$

B) $\alpha+\beta=7/3$, $\alpha\beta=2$

C) $\alpha+\beta=7/3$, $\alpha\beta=-2$

D) $\alpha+\beta=-7/3$, $\alpha\beta=2$

Answer: $\alpha+\beta=7/3$, $\alpha\beta=-2$

By Vieta's relations, $\alpha+\beta=-b/a=7/3$ and $\alpha\beta=c/a=-6/3=-2$.

4. Which monic equation has roots -2 and 5? [1 marks]

A) $x^2+3x-10=0$

B) $x^2-3x-10=0$

C) $x^2-7x+10=0$

D) $x^2+7x-10=0$

Answer: $x^2-3x-10=0$

The sum is 3 and the product is -10, so the equation is $x^2-3x-10=0$.

5. If the roots of $x^2+5x+6=0$ are real, what are their signs? [1 marks]

A) Both positive

B) Both negative

C) Opposite signs

D) One is zero

Answer: Both negative

Their product is positive, so the signs agree; their sum is -5, so both are negative.

6. Which is the correct factorisation of $2x^2-x-6$? [1 marks]

A) $(2x-3)(x+2)$
B) $(2x+3)(x-2)$
C) $(2x-1)(x+6)$
D) $(x-3)(2x+2)$

Answer: $(2x+3)(x-2)$

Expanding gives $(2x+3)(x-2)=2x^2-4x+3x-6=2x^2-x-6$.

7. True or false: if the discriminant of a quadratic equation is zero, it has two equal real roots. [1 marks]

□ True □ False

Answer: True

When $D=0$, the $\pm\sqrt{D}$ term vanishes in the quadratic formula, giving the repeated root $-b/(2a)$.

8. True or false: if the product of the roots of a quadratic is positive, both roots must be positive. [1 marks]

□ True □ False

Answer: False

A positive product only means the signs agree; the sign of the sum distinguishes two positive roots from two negative roots.

9. Solve $2x^2+x-3=0$ using the quadratic formula, giving both roots in simplest form. [3 marks]

Answer: $x=1$ or $x=-3/2$

Here $a=2$, $b=1$ and $c=-3$, so $D=25$ and $x=(-1\pm5)/4$, giving 1 and $-3/2$.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

10. If α and β are the roots of $x^2-4x-1=0$, find $\alpha^2+\beta^2$ without solving the equation. [3 marks]

Answer: 18

$\alpha+\beta=4$ and $\alpha\beta=-1$, so $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=16+2=18$.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

11. Form a quadratic equation with integer coefficients whose roots are 1/2 and -1/3. [3 marks]

Answer: $6x^2-x-1=0$

Start with $(x-1/2)(x+1/3)=0$ and multiply by 6 to obtain $6x^2-x-1=0$.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

12. Find the values of m for which the roots of $x^2+(m-2)x-3m=0$ are real and have opposite signs. [3 marks]

Answer: $m>0$

The product is $-3m$, so opposite signs require $-3m<0$, hence $m>0$. Then $D=(m-2)^2+12m=(m+4)^2>0$.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

Challenge and review

1. Derive the quadratic formula for $ax^2+bx+c=0$, where a is nonzero, by completing the square.

Divide by a: $x^2+(b/a)x=-c/a$. Add $(b/2a)^2$ to both sides to get $(x+b/2a)^2=(b^2-4ac)/(4a^2)$, hence $x=(-b\pm\sqrt{b^2-4ac})/(2a)$.

2. If α and β are the roots of $2x^2-5x-3=0$, form an equation whose roots are α+1 and β+1 without first finding α and β.

We have $\alpha+\beta=5/2$ and $\alpha\beta=-3/2$. The new sum is 9/2 and the new product is $(\alpha+1)(\beta+1)=2$, so $2x^2-9x+4=0$.

3. Error analysis: a student claims $D>0$ proves that both roots are positive. Explain what the discriminant actually proves and what additional information determines the signs.

$D>0$ proves only that the roots are distinct and real. Then use $\alpha\beta=c/a$: a negative product gives opposite signs; for a positive product, use $\alpha+\beta=-b/a$ to distinguish two positive from two negative roots.

Follow-up plan

Below 14/20: revisit the quadratic formula, discriminant, root relations and signs, then retry Questions 2-6 and 9-12. At 14 or above: continue to the online module challenge.