# Recurrence and Induction

Teacher guide and model answers

Chapter 2 | Module 10 | Student Book pages 91-105 | Suggested time: 50 minutes | 20 marks

## Learning goals

- Explain and apply recurrence evaluation with valid conditions and clear reasoning.
- Explain and apply linear recurrence transformation with valid conditions and clear reasoning.
- Explain and apply induction step with valid conditions and clear reasoning.

## Quick reference

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|---|---|
| **Recurrence evaluation** | A recurrence defines later terms from earlier terms plus enough initial data. |
| **Linear recurrence transformation** | A recurrence $a_{n+1}=pa_n+c$ can be shifted by a fixed point to become geometric. |
| **Induction step** | The induction step proves that truth at k forces truth at k+1. |

## Worked example

Given $u_1=3$ and $u_{n+1}=u_n+4$, find $u_{12}$. Worked solution: The recurrence adds 4 exactly 11 times: $u_{12}=3+11\cdot4=47$.

# Graduated practice

Answer without viewing the key. Show complete working for short-response questions.

**1. If $u_1=2$ and $u_{(n+1)}=3u_n-1$, find $u_4$. [1 marks]**

A) 14
B) 29
C) 41
D) 53

**Answer: 41**

$u_2=5$, then $u_3=14$, then $u_4=41$.

**2. If $a_1=5$ and $a_{(n+1)}=a_n+4$, what is the explicit form of $a_n$? [1 marks]**

A) $a_n=4n+1$
B) $a_n=5n+4$
C) $a_n=4n+5$
D) $a_n=5+4n$

**Answer: $a_n=4n+1$**

The sequence is arithmetic with first term 5 and difference 4, so $a_n=5+4(n-1)=4n+1$.

**3. If $a_1=1$ and $a_{(n+1)}=2a_n+3$, what is $a_4$? [1 marks]**

A) 21
B) 27
C) 29
D) 32

**Answer: 29**

The fixed point is -3. Let $b_n=a_n+3$; then $b_{(n+1)}=2b_n$ and $b_1=4$, so $a_4=4\times2^3-3=29$.

**4. To prove $1+3+5+...+(2n-1)=n^2$ by induction from $n=1$, which is the correct base case? [1 marks]**

A) $1=1^2$
B) $1+3=2^2$
C) $2n-1=n^2$
D) Assume the result at $n=1$

**Answer: $1=1^2$**

Substitute $n=1$ into both sides: the left side is 1 and the right side is $1^2=1$.

**5. In proving $1+2+...+n=n(n+1)/2$, assume $P(k)$. What expression results after adding $k+1$? [1 marks]**

A) $k(k+1)/2$
B) $(k+1)^2/2$
C) $(k+1)(k+2)/2$
D) $k(k+2)/2$

**Answer: $(k+1)(k+2)/2$**

$k(k+1)/2+(k+1)=(k+1)(k+2)/2$, which is the required form for $k+1$.

**6. To prove that $5^n-1$ is divisible by 4, which rearrangement supports the induction step? [1 marks]**

A) $5^{(k+1)}-1=5(5^k-1)+4$
B) $5^{(k+1)}-1=5^k+4$
C) $5^{(k+1)}-1=4(5^k-1)$
D) $5^{(k+1)}-1=5(5^k+1)$

**Answer: $5^{(k+1)}-1=5(5^k-1)+4$**

By the induction hypothesis, $5(5^k-1)$ is divisible by 4, and 4 is divisible by 4.

**7. True or false: a valid base case and proof that P(k) implies P(k+1) establish P(n) for every integer from the starting value onward. [1 marks]**

□ True □ False

**Answer: True**

The base case starts the chain, and the induction step carries truth from each case to the next.

**8. True or false: checking the first 100 cases of a statement is an induction proof for all natural numbers. [1 marks]**

□ True □ False

**Answer: False**

A finite check cannot establish infinitely many cases; a valid base case and induction step are required.

**9. If $u_1=1$, $u_2=3$ and $u_{(n+2)}=u_{(n+1)}+2u_n$, find $u_6$ and show the intermediate terms. [3 marks]**

**Answer: $u_6=43$**

$u_3=5$, $u_4=11$, $u_5=21$, and $u_6=21+2\times11=43$.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

**10. If $a_1=3$ and $a_{(n+1)}=a_n+2n$, derive an explicit form for $a_n$ and then find $a_8$. [3 marks]**

**Answer: $a_n=n^2-n+3$ and $a_8=59$**

$a_n=3+2\Sigma$ from $k=1$ to $n-1$ of $k=3+n(n-1)=n^2-n+3$, so $a_8=64-8+3=59$.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

**11. Prove by induction that the sum of the first n odd numbers is $n^2$ for every $n\geq1$. [3 marks]**

**Answer: The statement is true for every $n\geq1$**

Base $n=1$: $1=1^2$. If the sum through $2k-1$ is $k^2$, adding $2k+1$ gives $k^2+2k+1=(k+1)^2$.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

**12. Prove by induction that $7^n-1$ is divisible by 6 for every $n\geq1$. [3 marks]**

**Answer: 6 divides $7^n-1$ for every $n\geq1$**

Base: $7-1=6$. If $7^k-1=6m$, then $7^{(k+1)}-1=7(7^k-1)+6=6(7m+1)$, so it is divisible by 6.

Rubric: valid method 1 | correct substitution/calculation 1 | clear checked result 1

# Challenge and review

**1. Solve a1=1, a_(n+1)=2a_n+3 by a suitable shift and write an explicit formula for a_n.**

> The fixed point is -3. Let b_n=a_n+3; then b_(n+1)=2b_n and b1=4, so b_n=4×2^(n-1) and a_n=4×2^(n-1)-3.

**2. Prove by induction that 2^n≥n+1 for every n≥0.**

> Base n=0: 1≥1. If 2^k≥k+1, then 2^(k+1)=2×2^k≥2(k+1)≥k+2 because k≥0.

**3. Error analysis: in the induction step, a student assumes P(k+1) and rearranges it until reaching P(k+1). Explain why the proof is circular.**

> Only P(k) may be assumed before deriving P(k+1). Assuming the target statement makes the argument circular and supplies no independent reason for its truth.

## Follow-up plan

Below 14/20: revisit initial data, fixed-point shifts, and the base and induction steps, then retry Questions 3-6 and 9-12. At 14 or above: continue to the online module challenge.